

# A NOTE ABOUT THE $\{K_i(z)\}_{i=1}^{\infty}$ FUNCTIONS

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In the article [10], A. Petojević verified useful properties of the  $K_i(z)$  functions which generalize Kurepa's [1] left factorial function. In this note, we present simplified proofs of two of these results and we answer the open question stated in [10]. Finally, we discuss the differential transcendency of the  $K_i(z)$  functions.

A. PETOJEVIĆ [7, p. 3.] considered the family of functions:

$$(1) \quad {}_vM_m(s; a, z) = \sum_{k=1}^v (-1)^{k-1} \binom{z+m+1-k}{m+1} \mathcal{L}[s; {}_2F_1(a, k-z, m+2; 1-t)],$$

for  $\Re(z) > v-m-2$ , where  $v \in \mathbf{N}$  is a positive integer;  $m \in \{-1, 0, 1, 2, \dots\}$  is an integer;  $s, a, z$  are complex variables;  $\mathcal{L}[s; F(t)]$  is LAPLACE transform and  ${}_2F_1(a, b, c; x)$  is the hypergeometric function ( $|x| < 1$ ). Đ. KUREPA has considered in the articles [1, p. 151.] and [2, p. 297.] a complex function defined by the integral:

$$(2) \quad K(z) = \int_0^{\infty} e^{-t} \frac{t^z - 1}{t - 1} dt,$$

for  $\Re(z) > 0$ . Especially, for KUREPA's function  $K(z)$ , it is true that  $K(z) = {}_1M_0(1; 1, z)$ , for  $\Re(z) > 0$ , according to [10]. For various of values of parameters  $v, m, s, a, z$  from (1), different special functions, as presented in [10], are obtained. A. PETOJEVIĆ has considered in the article [10, p. 1640.] the following sequence of functions:

$$(3) \quad K_i(z) = \frac{{}_1M_0(1; 1, z+i-1) - {}_1M_0(1; 1, i-1)}{{}_1M_{-1}(1; 1, i)},$$

for  $i \in \mathbf{N}$  and  $\Re(z) > -i$ . On the basis of the definition in (3), the following representation via KUREPA's function is true:

$$(4) \quad K_i(z) = \frac{1}{(i-1)!} \left( K(z+i-1) - K(i-1) \right),$$

for  $i \in \mathbf{N}$  and  $\Re(z) > -i+1$ . Note that  $K(0)=0$  [2, p. 297.] and therefore  $K_1(z)=K(z)$  for  $\Re(z) > 0$ . Analytical and differential-algebraic properties of KUREPA's function  $K(z)$  are considered in articles [1–12] and in many other articles. *On the basis of well-known statements for KUREPA's function  $K(z)$ , using representation (4), in many cases we can get simple proofs for analogous statements for  $K_i(z)$  functions.* For example, it is a well-known fact that it is possible to analytically continue KUREPA's function to a meromorphic function with simple poles at integer points  $z = -1$  and  $z = -m$ , ( $m \geq 3$ ) [2, p. 303.], [3, p. 474.]. Residues of KUREPA's function at these poles have the following form [2]:

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$$(5) \quad \operatorname{res}_{z=-1} K(z) = -1 \quad \text{and} \quad \operatorname{res}_{z=-m} K(z) = \sum_{k=2}^{m-1} \frac{(-1)^{k-1}}{k!}, \quad (m \geq 3).$$

For KUREPA's function  $K(z)$  the infinite point is an essential singularity [3]. Hence, on the basis of (4), each function  $K_i(z)$  is meromorphic with simple poles at integer points  $z = -i$  and  $z = -(i+m)$ , ( $m \geq 2$ ). On the basis of (4) we have:

$$(6) \quad \operatorname{res}_{z=-(i+m)} K_i(z) = \frac{1}{(i-1)!} \cdot \operatorname{res}_{z=-(i+m)} K(z+i-1) = \frac{1}{(i-1)!} \cdot \operatorname{res}_{z=-(m+1)} K(z),$$

where  $m = 0$  or  $m \geq 2$ . Hence:

$$(7) \quad \operatorname{res}_{z=-i} K_i(z) = -\frac{1}{(i-1)!} \quad \text{and} \quad \operatorname{res}_{z=-(i+m)} K_i(z) = \frac{1}{(i-1)!} \cdot \sum_{k=2}^m \frac{(-1)^{k-1}}{k!}, \quad (m \geq 2).$$

For each  $K_i(z)$  function the infinite point is an essential singularity. Therefore, we get Theorem 3.3. from [10]. Next, it is a well-known fact that for KUREPA's function the following asymptotic relation  $K(x) \sim \Gamma(x)$  is true for real  $x$  such that  $x \rightarrow \infty$  and where  $\Gamma(x)$  is the gamma function [2, p. 299.]. Hence, for fixed  $i \in \mathbf{N}$  and real  $x > -i+1$ , on the basis of (4), we get:

$$(8) \quad \frac{K_i(x)}{\Gamma(x+i-1)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{\Gamma(x+i-1)} \xrightarrow{x \rightarrow \infty} \frac{1}{(i-1)!}$$

and

$$(9) \quad \frac{K_i(x)}{\Gamma(x+i)} = \frac{1}{(i-1)!} \cdot \frac{K(i+x-1) - K(i-1)}{(x+i-1)\Gamma(x+i-1)} \xrightarrow{x \rightarrow \infty} 0.$$

Therefore, we get Theorem 3.6. from [10]. Next we give a solution to the open problem stated in Question 3.7. in [10]. Namely, the following formula in the article [8, p. 35.] is given:

$$(10) \quad K(z) = \frac{\operatorname{Ei}(1) + i\pi}{e} + \frac{(-1)^z \Gamma(1+z) \Gamma(-z, -1)}{e},$$

for values  $z \in \mathbf{C} \setminus \{-1, -2, -3, -4, \dots\}$  and  $i = \sqrt{-1}$ . In the previous formula  $\operatorname{Ei}(z)$  and  $\Gamma(z, a)$  are exponential integral and incomplete gamma function respectively [8]. Then, for fixed  $i \in \mathbf{N}$  and values  $z \in \mathbf{C} \setminus \{-i, -i-1, -i-2, -i-3, \dots\}$ , on the basis of (4) and (10), we get:

$$(11) \quad \begin{aligned} K_i(z) &= \frac{1}{(i-1)!} \left( K(z+i-1) - K(i-1) \right) \\ &= \frac{\operatorname{Ei}(1) + i\pi}{e(i-1)!} + \frac{(-1)^{z+i-1} \Gamma(1+z+i-1) \Gamma(-z-i+1, -1)}{e(i-1)!} \\ &\quad - \frac{\operatorname{Ei}(1) + i\pi}{e(i-1)!} - \frac{(-1)^{i-1} \Gamma(i) \Gamma(-i+1, -1)}{e(i-1)!} \\ &= (-1)^i e^{-1} \left( \Gamma(1-i, -1) - (-1)^z \frac{\Gamma(1-i-z, -1) \Gamma(i+z)}{(i-1)!} \right). \end{aligned}$$

Therefore, the affirmative answer for Question 3.7. from [10] is true for complex values  $z \in \mathbf{C} \setminus \{-i, -i-1, -i-2, -i-3, \dots\}$ .

Finally, at the end of this note let us emphasize one differential–algebraic fact for the sequence of functions  $K_i(z)$ . On the basis of the formula (17) from the article [10], we can conclude that each  $K_i(z)$  function satisfies the following recurrence relation  $(i-1)!K_i(z+1) - (i-1)!K_i(z) = \Gamma(z+i)$ . The previous relation can be used to verify the differential transcendency of these functions as discussed in [11, 12]. Therefore, we can conclude that each  $K_i(z)$  function is a differential transcendental function, i.e. it satisfies no algebraic differential equation over the field of complex rational functions.

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